# Modeling and control of a magnetic levitation system

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## I. INTRODUCTION

The system under study is a an electromagnet that generates an attractive force to suspend a metal ball. An electromagnet generates magnetomotive force (MMF) when current flows through the coils wound around its ferroelectric core. The objective is to maintain the position of the ball by controlling the MMF generated by regulating the current through the coils. This system is inherently unstable and demands a controller to ensure stability, making it a common case-study for various kinds of control approach. Fig.1 shows the system under study.

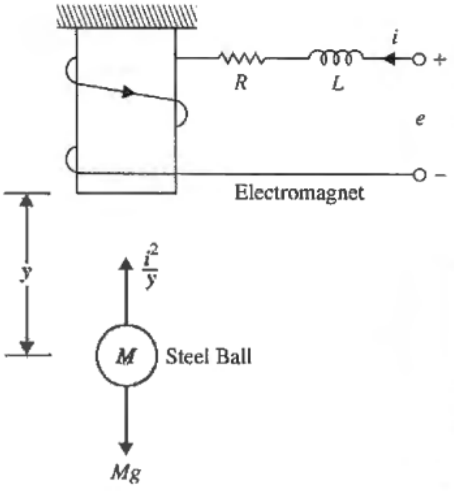

Fig. 1. Magnetically suspended metal ball [1]

## II. SYSTEM DESCRIPTION AND LINEAR MODEL

This system can be separated into two parts, namely, the electrical subsystem that drives the current in the coils of the electromagnet, and the mechanical subsystem concerned with the electromagnet and ball dynamics [2]. Let $e$ denote the input voltage, $y$ denote the position of the ball, $i$ be the current, $R$ is the load resistance, $L$ the lumped coil inductance, $M$ be the mass of the ball, and $g$ denote gravity. From Kirchoff's voltage law applied to the electrical circuit,

$$e = Ri + L\frac{di}{dt} \tag{1}$$

From Newton's second law of motion, the sum of all forces acting on an object is zero.

$$M\frac{d^2y}{dt^2} = Mg - \frac{i^2}{y}, \quad y \neq 0 \tag{2}$$

Let the state variables of the system be $x = [i, y, \dot{y}]'$, where $\dot{y}$ denotes the velocity of the ball. The state equations can be formed as,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{L}\left(e - Rx_1\right) \\ x_3 \\ g - \frac{1}{M}\frac{x_1^2}{x_2} \end{bmatrix} \tag{3}$$

The output variable can be chosen to be the position of the ball from the electromagnet, i.e, y.

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x \tag{4}$$

Thus, the nonlinear system model is written in $\dot{x} = f(x, u_{in})$, $y = g(x, u_{in})$ where $u_{in}$ is the input voltage $e$ to the electromagnet. Thus, the system has $n = 3$ states.

The control objective here is to control the position of the ball by regulating the voltage $e$ which in turn adjusts the coil current $i$. Consider the point in time $t = t_o$ where the ball is at a nonzero distance away from the electromagnet, $y(t_0) = h_0$. This is an equilibrium point for the mechanical subsystem since,

$$y = h_o \quad (constant) \tag{5}$$

$$\dot{x_2} = x_3 = 0 \tag{6}$$

It follows that $\ddot{y} = 0$, so the nominal value of current can be found using (2),

$$i = \sqrt{Mgh_o}, \quad i \in \mathbb{R}_{>0} \tag{7}$$

The nonlinear system can be linearized about the equilibrium point $(x_{eq}, u_{eq}) = ([\sqrt{Mgh_o} \quad h_o \quad 0]', R\sqrt{Mgh_o})$. Hence $A$ matrix can be formed by taking the Jacobian where $A_{ij} = \frac{\partial \dot{f}_i}{\partial x_j}$. Equations (8) - (11) are obtained as follows,

$$A = \begin{bmatrix} \frac{-R}{L} & 0 & 0 \\ 0 & 0 & 1 \\ -2\sqrt{\frac{g}{Mh_0}} & \frac{g}{h_0} & 0 \end{bmatrix} \tag{8}$$

$$B = \begin{bmatrix} \frac{1}{L} & 0 & 0 \end{bmatrix}^T \tag{9}$$

$$C = [0 \quad 1 \quad 0] \tag{10}$$

$$D = 0 \tag{11}$$

The linearized system can now be written in the standard state space form as

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{12}$$

The properties of this LTI system and possible controllers are studied in the following sections.

TABLE I
PHYSICAL PARAMETERS OF LTI SYSTEM

| *Parameter* | *Symbol* | *Value* |
|---|---|---|
| Resistance | R | 5 $\Omega$ |
| Inductance | L | 100 $mH$ |
| Gravitational acceleration constant | g | 9.8 $m/s^2$ |
| Mass | m | 1 $kg$ |
| Height | $h_o$ | 0.5 $m$ |

## III. SYSTEM ANALYSIS

### *A. Stability*

The physical parameters of the LTI system are listed in Table I. The stability of the system can be determined by finding the eigenvalues of the matrix A. If all eigenvalues have $Re(\lambda) < 0$, the system is globally asymptotically stable and Lyapunov stable. However, if there are eigenvalues with $Re(\lambda) > 0$, the system is unstable. The system may be stable if the eigenvalues have $Re(\lambda) = 0$ if the corresponding Jordan blocks are scalars. The eigenvalues of the A matrix are $\lambda_1 = 4.4272, \lambda_2 = -4.4272, \lambda_3 = -50$. Since there is one eigenvalue $Re(\lambda_1) > 0$, the system is unstable. The zero-input LTI system is simulated and it is indeed unstable as in Fig.2 the output goes to infinity with time and does not converge.

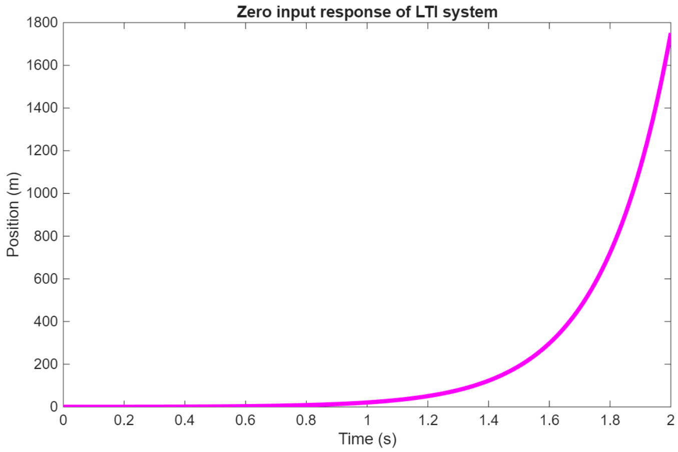

Fig. 2. Simulated zero input response of modelled system

### *State transition matrix*

Another way to show the instability is to calculate the zero-input response of the system $\dot{x} = Ax, x(0) = x_0$ given by $x(t) = e^{At}x_0$ where $e^{At}$ describes the evolution of the states. The matrix exponential is calculated as $e^{At} = L^{-1}[(sI-A)^{-1}]$ is shown in (13). The presence of a non-zero matrix coefficient of $e^{4.4272t}$ makes the system unstable.

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & -0.0035 & 0.1785 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{-50t} \\ &+ \begin{bmatrix} 0 & 0.0219 & -0.097 \\ 0 & 0.1129 & -2.213 \\ 0 & -0.1129 & 0.5 \end{bmatrix} e^{-4.427t} \\ &+ \begin{bmatrix} 0 & -0.018 & -0.081 \\ 0 & 0.1129 & -2.213 \\ 0 & -0.1129 & 0.5 \end{bmatrix} e^{4.427t} \end{aligned} \tag{13}$$

### *B. Controllability*

The concept of controllability is to determine if inputs to the system can drive it to any value in the state space in finite time. Formally, an LTI This property of a system can be determined using the controllability matrix test that states that an LTI system is controllable if and only if rank($\mathcal{C}$) = n, where $\mathcal{C}$ is given by

$$\mathcal{C} = [B \ AB \ ...A^{n-1}B] \tag{14}$$

Using MATLAB, rank($\mathcal{C}$) $=$ 3 $=$ n. Thus, the system is controllable.

### *C. Stabilizability*

Stabilizability is a weaker notion than controllability. An (unstable) LTI system is potentially stabilizable if the state variables that are not controllable are asymptotically stable. The PBH test for stabilizability states that if rank($[\lambda I - AB]$) $= n$ for the eigenvalues with Re($\lambda$) $> 0$, the system is stabilizable. In MATLAB, this is evaluated for $\lambda_1 = 4.4272$ and the rank $= 3$. Thus, the linearized system is stabilizable.

### *D. Observability*

The concept of observability is to be able to infer the initial state $x(0)$ from future outputs and inputs $y(t), u(t)$ respectively, $t \in [0, T]$ for any $T \in \mathbb{R}_{>0}$. It only depends on (A, C) matrices. The observability matrix is obtained as a dual of the controllability matrix since (A,C) is observable if and only if $(A^T, C^T)$ is controllable, given as

$$\mathcal{O} = [C \ CA \ ...CA^{n-1}]^T \tag{15}$$

The observability of the LTI system depends on the variable defined as the output. There may be some state(s) that may make the system unobservable if chosen as the output. In this study, the state ball position is chosen as the output, thus $C = [0 \ 1 \ 0]$ as defined in section II. The $rank(\mathcal{O}) = 3 = n$. This system is observable. However, if the output was chosen to be the coil current, giving $C = [1 \ 0 \ 0]$, $rank(\mathcal{O}) = 1$, making the system unobservable. This implies that the states cannot be reconstructed by taking the coil current to the the output.

### *E. Observer design*

Since (A,C) is observable, the general realization of a closed-loop observer for an LTI system is adopted as,

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu \tag{16}$$

where L is the gain matrix of the full-state observer. The error (e) dynamics are given by

$$\dot{e} = (A - LC)e, \quad e(0) = x(0) \tag{17}$$

If all the eigenvalues of A-LC lie in the left half plane, the error dynamics can be made asymptotically stable. By rule of thumb, the observer dynamics should be atleast 5 times faster than the plant dynamics. The observer poles are chosen to be placed at $s = -60$, $s = -50$, $s = -40$, giving a gain matrix

$$L = [1e{-}13 \quad 100 \quad 2.5e3] \tag{18}$$

Fig.3 shows the observed state responses compared to the real states where all three converge in less than 0.4 seconds and track the real states well at steady state despite $\hat{x}_2$ and $\hat{x}_3$ having large overshoots.

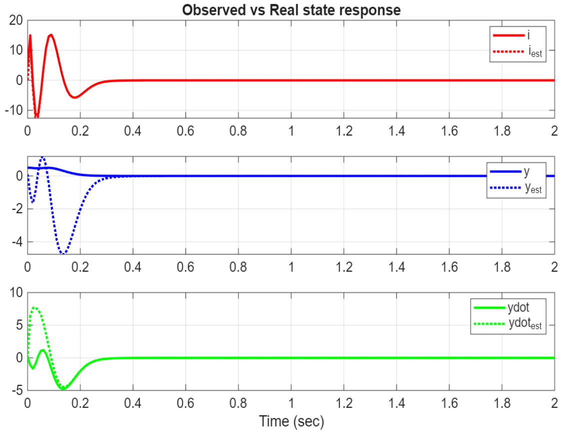

Fig. 3. System state responses compared to observed state responses

## IV. CONTROL APPROACHES

The control objective studied here is to maintain the ball position. The transfer function relating the ball position to the voltage input is given by (19)

$$H(s) = \frac{Y(s)}{U(s)} = \frac{-88.54}{s^3 + 50s^2 - 19.6s - 980} \tag{19}$$

Pole-placement design through state feedback and linear quadratic control is studied here. The system representation in Controllable Canonical Form (CCF) shows the system's inherent controllability clearly making it ideal for pole placement. From (19), the CCF is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 980 & 19.6 & -50 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{20}$$

### *A. State feedback control*

For full-state feedback control, all three state variables must be fed back to the controller, i.e, $u(t) = -Kx(t) + r(t)$ where $K$ is a $1 \times n$ feedback gain matrix and $r(t)$ is the reference input. Since (A,B) is controllable, the eigenvalues of the new system given by $\dot{x} = (A - BK)x(t) + Br(t)$ can be arbitrarily placed. Let the design specifications be

- System must be stable
- For any disturbance on the position of the ball from its equilibrium point the ball must return to the equilibrium point within 1 second without steady state error
- System response to a disturbance must not have over 10% transient overshoot

The required characteristic polynomial can be constructed by multiplying a second order transfer function with a third pole (appendix). The design criterion implies a damping constant $\zeta = 0.8066$, $w_n = 5$ rad/s. Two poles are the complex pole pair $s = -4.033 \pm j2.955$ . The third pole of the new system is further away at $s = -25$. The required characteristic polynomial is

$$s^3 + 33.07s^2 + 226.6s + 624.9 \tag{21}$$

The corresponding gain matrix for state feedback is

$$K = [-1.693 \quad -14.37 \quad -2.78] \tag{22}$$

The new system eigenvalues are located at

$$\lambda = \begin{bmatrix} -16.0208 \\ -4.026 + j2.9655 \\ -4.0246 - j2.9655 \end{bmatrix} \tag{23}$$

Controller 2 places the poles further to the left of the origin, at $-20 \pm j20, -100$ is designed to see the difference in the response. Fig.4 shows the zero input impulse response of the two closed loop systems.

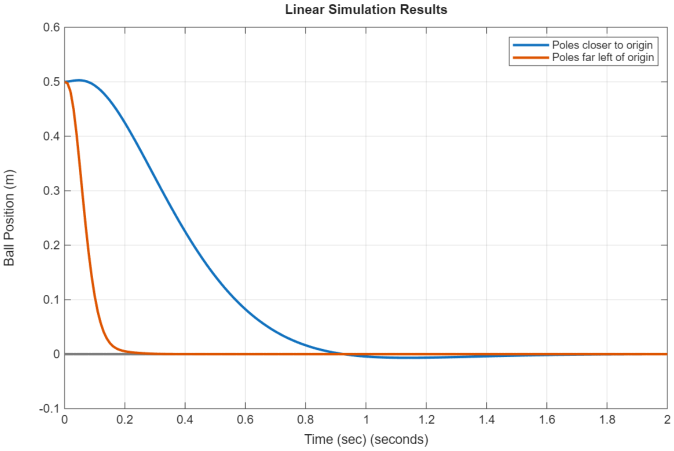

Fig. 4. Zero input response of closed loop system with two different state-feedback controllers

### *B. Linear Quadratic Regulator (LQR)*

An alternate control approach is taken to maintain the position of the ball through the voltage input. The general formulation of an optimal control problem for an LTI system $\dot{x} = Ax + Bu$ finds the optimal inputs to minimize the infinite-horizon cost function given by [3]

$$J = \int_0^\infty [x^TQx + u^TRu]dt, \quad Q = Q^T \succeq 0, R = R^T \succ 0 \tag{24}$$

where $Q$ and $R$ are weights imposed on tracking error and control effort, respectively. $Q$ is positive semi-definite since all states do not need to be penalized, but $R$ is positive definite to penalize all inputs used. This gives a solution of the form

$$u = -R^{-1}B^TPx \tag{25}$$

where $P = P^T$ is the solution to the algebraic Riccati equation

$$A^TP + PA + Q - PBR^{-1}B^TP = 0 \tag{26}$$

This holds for the system under consideration, since it is stabilizable. A higher $Q$ should be chosen for critical states, and a higher $R$ for costly inputs. For this single-input system, different choices of $Q$ and $R$ are examined, as shown below,

| **Case 1** | **Case 2** | **Case 3** |
|---|---|---|
| $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $Q = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 100 \end{bmatrix}$ | $Q = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $R = 1$ | $R = 0.1$ | $R = 0.1$ |

*Note: This section examines state responses under different $Q$ and $R$ matrices but does not come up with these matrices for a specific design criteria. Weights may not be practically feasible.*

Figs.5 - 7 show the state responses to the closed-loop LQR systems for cases 1 - 3. Case 1 chooses $Q = C'C$, $R = 1$ to place equal importance on tracking error of the states that are the outputs and control effort. While case 1 shows least peak in control effort among the three cases, the settling time of $x_2$, the output, is nearly 1.4s. With higher weights at $Q(1,1)$ and $Q(3,3)$ in case 2, Fig.6 shows that $x_3$ has a smaller overshoot and the control effort is maximum. Low $Q(2,2)$ weight on $x_2$ reflects as a longer settling time of $> 2s$. Compared to case 1, a higher $Q(2,2)$ yields a shorter settling time of 0.8s for $x_2$ at the cost of higher control effort.

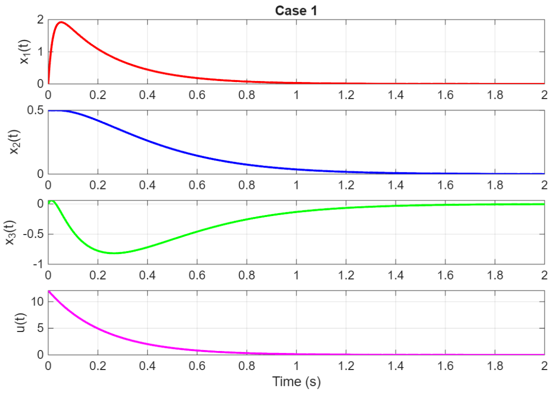

Fig. 5. Case 1: State trajectories (red, blue, green) and control effort (pink)

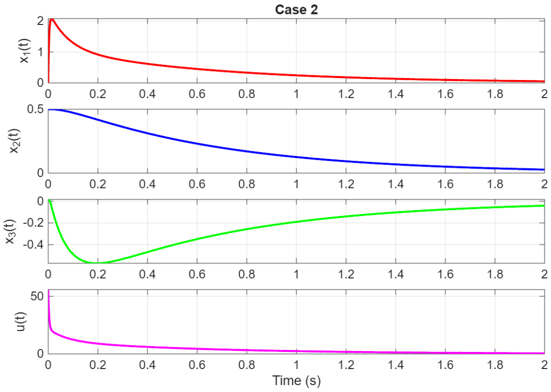

Fig. 6. Case 2: State trajectories (red, blue, green) and control effort (pink)

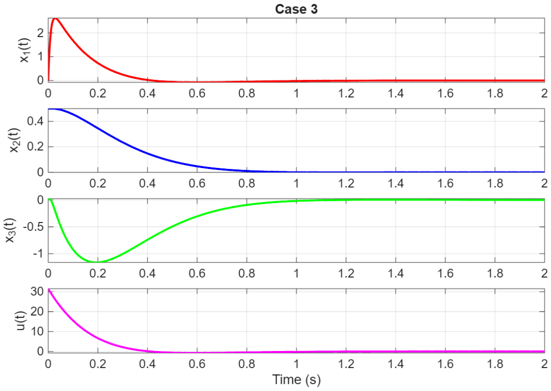

Fig. 7. Case 3: State trajectories (red, blue, green) and control effort (pink)

## V. CONCLUSION

A nonlinear system is modeled and linearized about an equilibrium point where there may be many. The system properties are studied systematically with the intention of designing a suitable controller. Firstly, the system is seen to be unstable yet stabilizable. Moreover, it is fully controllable. This is evident mathematically by successfully representing the system in its CCF. The linearized system is also fully observable, and an example observer design is shown. Two approaches to controller design are studied. Although the pole-placement approach seems to be more ”design” oriented, LQR sets the optimization problem where the ”design” is in choosing $Q$ and $R$ weight matrices, and these are illustrated through simulations. Interestingly, the LQR yields an input matrix that resembles a state feedback controller.

## REFERENCES

[1] BC Kuo. *Automatic Control Systems*. John Wiley and Sons, Inc., 2009.

[2] Emani-Naemi Franklin Powell. *Feedback Control of Dynamic Systems*. Pearson Education Ltd., 2020.

[3] João.P.Hespanha. *Linear Systems Theory*. Princeton University Press, 2010.

## VI. APPENDIX

### *Dominant pole approximation*

Without loss of generality, let a single-input single output system with three states have the control-to-output transfer function as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\alpha\omega_o^2}{(s+\alpha)(s^2+2\zeta\omega_o s+\omega_o^2)} \tag{27}$$

Then it follows that

$$G(s) \approx \begin{cases} \frac{\alpha}{s+\alpha}, & \alpha \ll \zeta w_o \\ \frac{w_o^2}{s^2+2\zeta w_o s+w_o^2}, & \alpha \gg \zeta w_o \end{cases} \tag{28}$$

From (28), a third order transfer function's desired transient response can be obtained by placing poles for the approximate second order transfer function, and placing the third pole far away to the left of the origin to ensure stability.